A REMARK ON THE BOSON-FERMION CORRESPONDENCE

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Abstract

We introduce the space of skew-symmetric functions depending on an infinite number of variables and give a simple interpretation of the boson-fermion correspondence.

The boson-fermion correspondence (Skyrme, 1971) is a canonical transformation from bosonic Fock space to fermionic Fock space (more precisely it is an operator from some special bosonic Fock space to some special fermionic Fock space). Now it is a quite well-known object in mathematics and mathematical physics (see for instance [3]). The purpose of this note is to give a very simple description of this operator. The boson-fermion correspondence will be multiplication with the Vandermonde determinant. In some sense our description is not new (it is equivalent to an explanation which uses Schur functions, see [3]), on the other side I never have seen this description in the literature and never have heard about it.

1. Bosonic Fock space F. Consider formal variables $z_1, z_2, \ldots$. Consider the space Pol of polynomials in the variables $z_1, z_2, \ldots$. Define a scalar product in Pol by the following rule: the monomials $z_1^{k_1} z_2^{k_2} \ldots$ are pairwise orthogonal and

$$\|z_1^{k_1} z_2^{k_2} \ldots\|^2 = \prod_j \left(k_j! j^{k_j}\right) . \tag{1}$$

We define the bosonic Fock space **F** (V.A.Fock, 1929, see [3, 2]) as the completion of Pol with respect to this scalar product.

2. Space Symm of symmetric functions. Consider an infinite collection of formal variables $x_1, x_2, \ldots$. We define the space **Symm** of symmetric functions as the space of symmetric infinite formal sums of monomials in the variables $x_1, x_2, \ldots$ (see [1]) (in each monomial only finite number of variables occurs).

Denote by p_k the infinite Newton sums

$$p_n = x_1^n + x_2^n + x_3^n + \ldots$$

The classical scalar product (J.H. Redfield, 1927) in the space **Symm** is given by the rule: the "functions" $p_1^{k_1} p_2^{k_2} \ldots$ are orthogonal and

$$\| p_1^{k_1} p_2^{k_2} \ldots \|^2 = \prod_j \left(k_j! j^{k_j} \right) . \tag{2}$$

3. Boson–symmetric correspondence, see [3]. A canonical isometry $I : \mathbf{F} \to$ **Symm** is given by the rule

$$I : z_1^{k_1} z_2^{k_2} \ldots \mapsto p_1^{k_1} p_2^{k_2} \ldots$$

In other words the operator I is a substitution operator

$$I f(x_1, x_2, x_3, \ldots) = f\left(\sum_j x_j, \sum_j x_j^2, \sum_j x_j^3, \ldots \right) .$$

Obviously I is an isometry (see (1) and (2)).

4. Space of skew-symmetric functions. This object is very simple but psychologically strange. Consider the same variables $x_1, x_2, \ldots$. A *quasi-monomial* is a formal expression

$$x_1^{\omega + l_1} x_2^{\omega + l_2} x_3^{\omega + l_3} \ldots$$

where $l_j = -j$ for large j and ω is a formal symbol. A *skew-symmetric function* is a formal (infinite) linear combination of quasi-monomials which is skew-symmetric with respect to all finite permutations of the variables $x_1, x_2, \ldots$.

REMARK. Informally, ω means

$$\omega = \infty .$$

It is "the total number" of variables $x_1, x_2, \ldots$. It is natural to consider the expression

$$\prod_{1 \leq i < j < \infty} (x_i - x_j) \tag{3}$$

as skew-symmetric function. Indeed let us write this expression in the form

$$\prod_{1 \leq i < j < \infty} \left\{ x_i \left(1 - \frac{x_j}{x_i} \right) \right\} = \prod_{1 \leq i < j < \infty} x_i \prod_{1 \leq i < j < \infty} \left(1 - \frac{x_j}{x_i} \right)$$

We obtain

$$\prod_{1 \leq i < j < \infty} (x_i - x_j) = \sum_{\sigma \in S_\infty} (-1)^\sigma x_1^{\omega - \sigma(1)} x_2^{\omega - \sigma(2)} \ldots \tag{4}$$

where S_∞ is the group of all finite permutations of the set $\{1, 2, 3, 4, \ldots\}$.

Let $l_1 < l_2 < l_3 < \ldots$ be integers and let $l_j = j$ for large j. Consider the basic skew-symmetric functions

$$S_{l_1, l_2, \ldots} = \sum_{\sigma \in S_\infty} (-1)^\sigma x_1^{\omega - l_{\sigma(1)}} x_2^{\omega - l_{\sigma(2)}} x_3^{\omega - l_{\sigma(3)}}$$

A scalar product in the space **Asymm** of skew-symmetric functions is defined by the rule: the functions $S_{l_1, l_2, \ldots}$ form an orthonormal basis in **Asymm**.

5. Correspondence between Symm and Asymm. A canonical isometry $J : \mathbf{Symm} \to \mathbf{Asymm}$ is given by the formula

$$J f(x_1, x_2, \ldots) = f(x_1, x_2, \ldots) \cdot \prod_{1 \leq i < j < \infty} (x_i - x_j) .$$

6. Fermionic Fock space, see [3, 2]. Let $\ldots \xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \xi_2, \ldots$ be a family of anticommuting variables ($\xi_i \xi_j = -\xi_j \xi_i$). Consider infinite products

$$\xi_{l_1} \xi_{l_2} \xi_{l_3} \cdots \tag{5}$$

where $l_1 < l_2 < \ldots$ and $l_j = j$ for large j. We define the *fermionic Fock space* $\boldsymbol{\Lambda}$ as the space where the monomials (5) form an orthonormal basis.

7. Isometry between Symm and $\boldsymbol{\Lambda}$. This correspondence is obvious: the basis element (4) corresponds to the basis element (5).

8. The boson-fermion correspondence is the composition of the correspondences

$$\mathbf{F} \to \mathbf{Symm} \to \mathbf{Asymm} \to \boldsymbol{\Lambda} \ .$$

In fact it is the composition of the substitution

$$z_k = \sum_j x_j^k$$

and the multiplication with the Vandermonde determinant (3).

Some additionsl discussion of boson-symmetric correspondences is contained in [4], [5]

References

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